Control of 1-D hyperbolic systems

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Outline

The hyperbolic control system considered

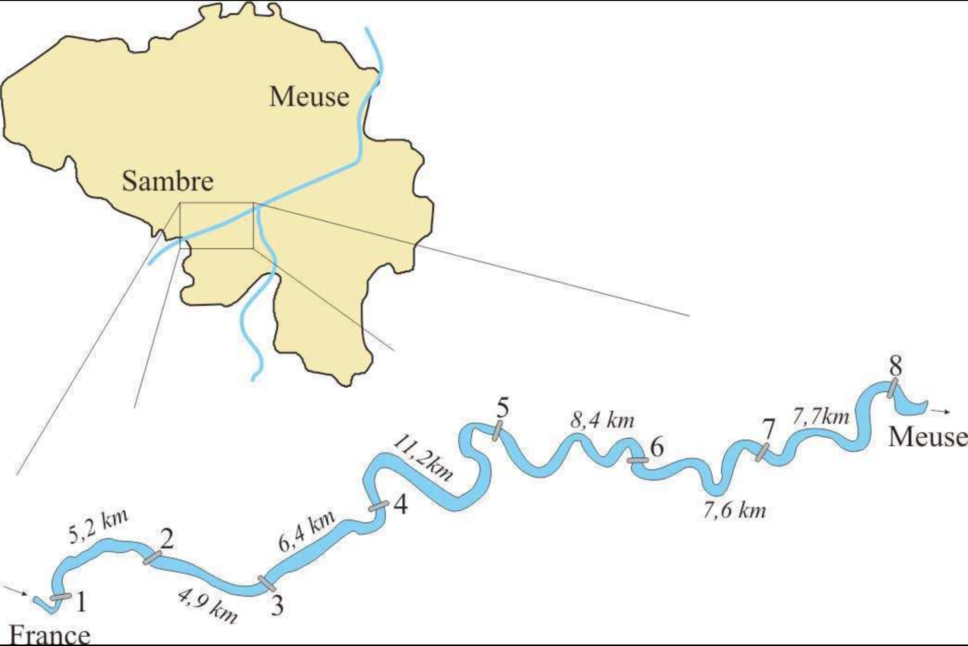
Our hyperbolic control system is

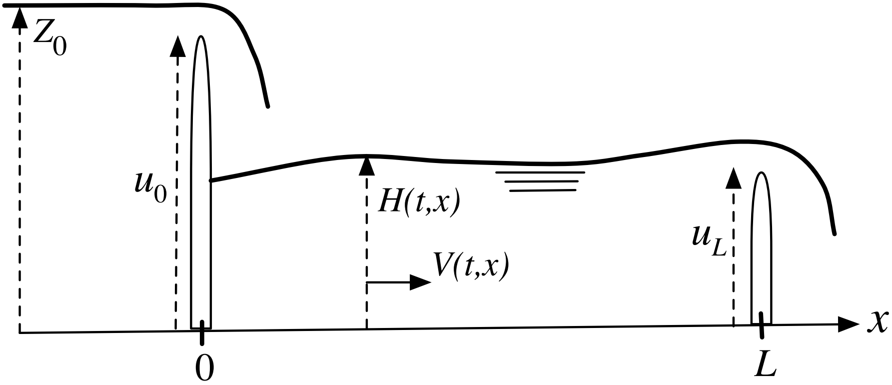
$$(1) \qquad y_t + A(y)y_x = 0, \; (t,x) \in [0,T] \times [0,L],$$

where, at time $t \in [0,T]$, the state is $x \in [0,L] \mapsto y(t,x) \in \mathbb{R}^n$. Let $y^* \in \mathbb{R}^n$ be fixed. Assume that $A(y^*)$ has n distinct real distinct eigenvalues: $\Lambda_1(y^*) < \ldots < \Lambda_m(y^*) < 0 < \ldots < \Lambda_n(y^*)$ for some $m \in \{0, \cdots, n\}$. After a linear change of variables, we mays assume that $A(y^*) = \text{diag}\,(\Lambda_1(y^*), \ldots, \Lambda_n(y^*))$. For $y \in \mathbb{R}^n$, let $y_- \in \mathbb{R}^m$ and $y_+ \in \mathbb{R}^{n-m}$ be such that

$$y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix}.$$

The control is $y_+(t,0)$ and $y_-(t,L)$.


Meuse
Sambre
5,2 km
2
1
France
4,9 km
3
6,4 km
4
11,2km
5
8,4 km
6
7,6 km
7
7,7km
8
Meuse



Z_0
u_0
$H(t,x)$
$V(t,x)$
u_L
x
0
L

The Saint-Venant equations

The index j is for the j-th reach.
Conservation of mass:

$$H_{jt} + (H_j V_j)_x = 0. \tag{1}$$

Conservation of momentum:

$$V_{jt} + \left(gH_j + \frac{V_j^2}{2} \right)_x = 0. \tag{2}$$

Flow rate: $Q_j = H_j V_j$.

Barré de Saint-Venant
(Adhémar-Jean-Claude)
1797-1886
Théorie du mouvement non permanent des eaux, avec applications aux crues des rivières et à l'introduction des marées dans leur lit, C. R. Acad. Sci. Paris Sér. I Math., vol. 53 (1871), pp.147–154.

Riemann coordinates

Let l be the number of reachs. Let $n = 2l$,

(1) $$y_i = V_i - \sqrt{2gH_i},\ y_{i+l} = V_i + \sqrt{2gH_i},\ \forall i \in \{1, \dots, l\},$$

(2) $$\Lambda_i = V_i - \sqrt{gH_i},\ \Lambda_{i+l} = V_i + \sqrt{gH_i},\ \forall i \in \{1, \dots, l\}.$$

Then the Saint-Venant equations can be written as

(3) $$y_t + A(y)y_x = 0,$$

with

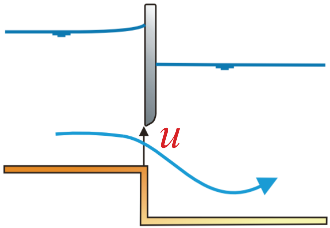
(4) $$A(y) := \text{diag}\ (\Lambda_1(y), \dots, \Lambda_n(y)).$$

The flow is subcritical flow $V_i < \sqrt{gH_i}$, $\forall i \in \{1, \dots l\}$. For subcritical flows, one has $m = l$.

La Sambre: Hydraulic gates

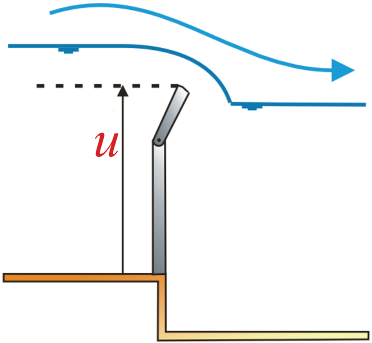
Boundary conditions

Underflow (sluice)



$$Q = K\sqrt{u(H_{up} - H_{down})}$$

Overflow (spillway)



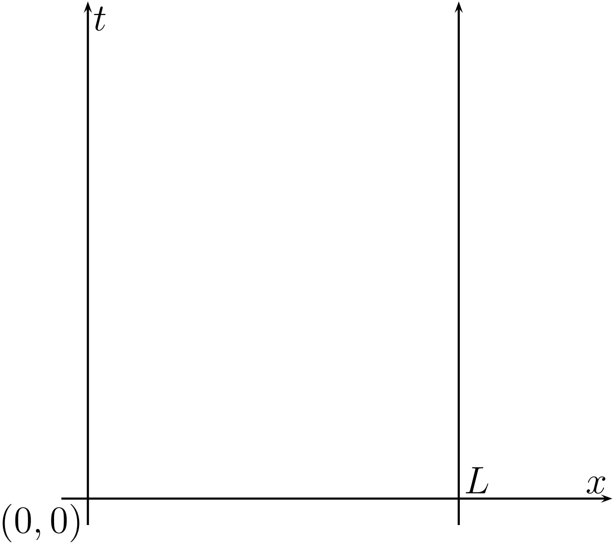
$$Q = K(H_{up} - u)^{3/2}$$

The control problem

Let $T > 0$. Given $y^0 : [0,1] \to \mathbb{R}^n$ and $y^1 : [0,1] \to \mathbb{R}^n$. Does there exist a solution of the control system such that $y(0,x) = y^0(x)$ and $y(T,x) = y^1(x)$? Local controllability: y^0 and y^1 are close to y^*.

A very simple example

$y_t + y_x = 0$, $y(t, 0) = u(t)$, $x \in [0, L]$, $t > 0$.



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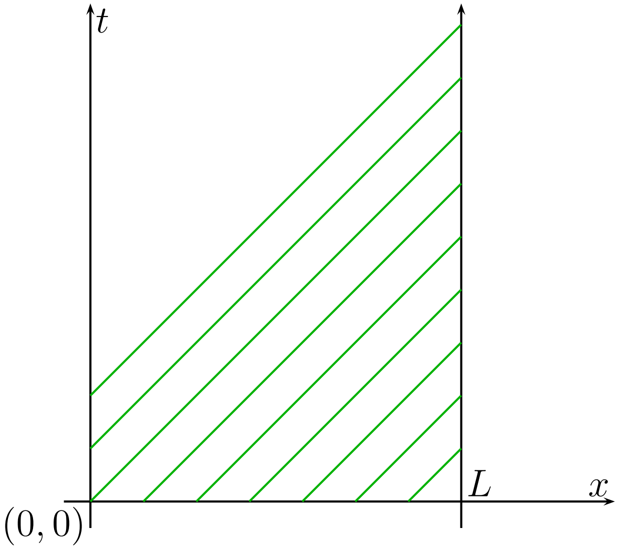

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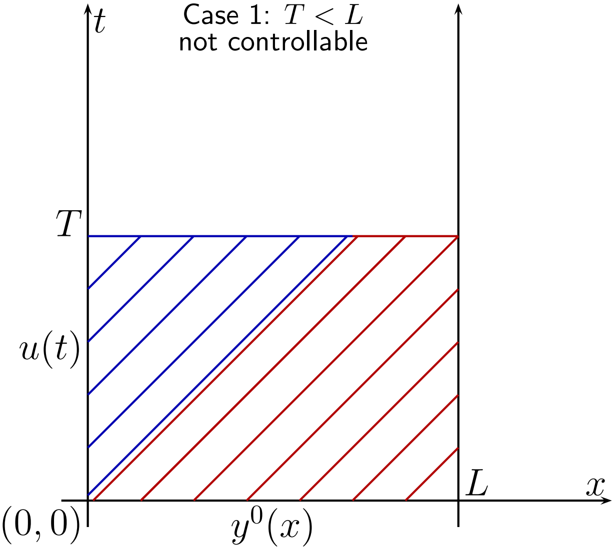
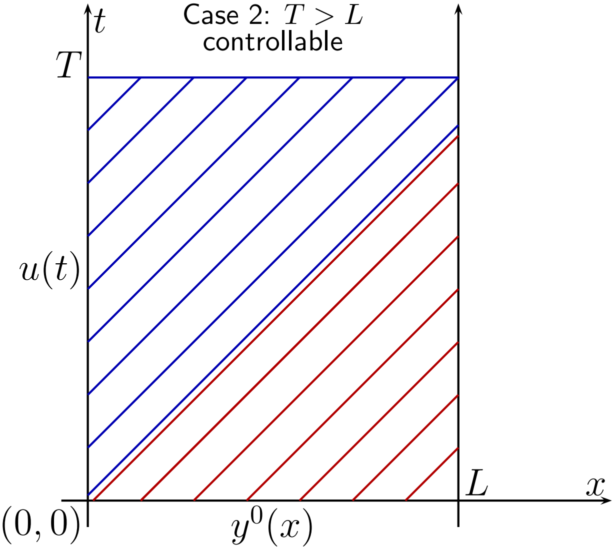


A very simple example

$y_t + y_x = 0,\ y(t,0) = u(t),\ x \in [0,L],\ t > 0.$



Controllability theorem

Theorem (Tatsien Li and Bopeng Rao (2003))

The local controllability for the C^1-norm holds if and only if $T > T_c$ with

$$T_c := \max\left\{\frac{L}{|\Lambda_1(y^*)|}, \ldots, \frac{L}{|\Lambda_n(y^*)|}\right\}. \tag{1}$$

Complements

- For the control on one side only, see T. Li and B. Rao (2002).
- Global steady states controllability for the Saint-Venant equations: M. Gugat (2003), M. Gugat and G. Leugering (2003, 2009). Friction and slopes are allowed in the last paper.
- Generalization: $A(t, x, y)$: Z. Wang (2007).

Complements: BV solutions

• F. Ancona and A. Marson (1998): $y_t + f(y)_x = 0$, $x \in (0, +\infty)$, $y(0, x) = 0$: reachable set.
• Th. Horsin (1998): $y_t + (y^2/2)_x = 0$, $x \in (0, L)$, $y(0, x) = y^0(x)$: approximate controllability under general conditions on the desired target.
• V. Perrollaz (2012): $y_t + (f(y))_x = u(t)$, $x \in (0, L)$, $y(0, x) = y^0(x)$: global controllability in small time.
• Adimurthi, S. Ghoshal and G. Gowda (2014): $y_t + (f(y))_x = 0$, $x \in (0, L)$, $y(0, x) = y^0(x)$: reachable set.
• A. Bressan and G. Coclite (2002): In the BV class of entropy solutions, already with $n = 2$, there are cases where one cannot steer the control system from y^0 to y^* even if y^0 is close to y^* (if one remains close to y^* in the BV-norm).
• O. Glass: One-dimensional Euler isentropic (2007) and non-isentropic system (2014), both in Eulerian and Lagrangian coordinates: Study of the controllability. Corollary: one can steer the control to y^* if y^0 is close (in the BV-norm) to y^* (while remaining close to y^* in the BV-norm).

Sketch of proof of Li-Rao's theorem

We assume that

$$(1) \qquad T > \max\{L/|\Lambda_i(y^*)|;\, i \in \{1, \dots, n\}\}.$$

Let $T_1 > 0$ be such that

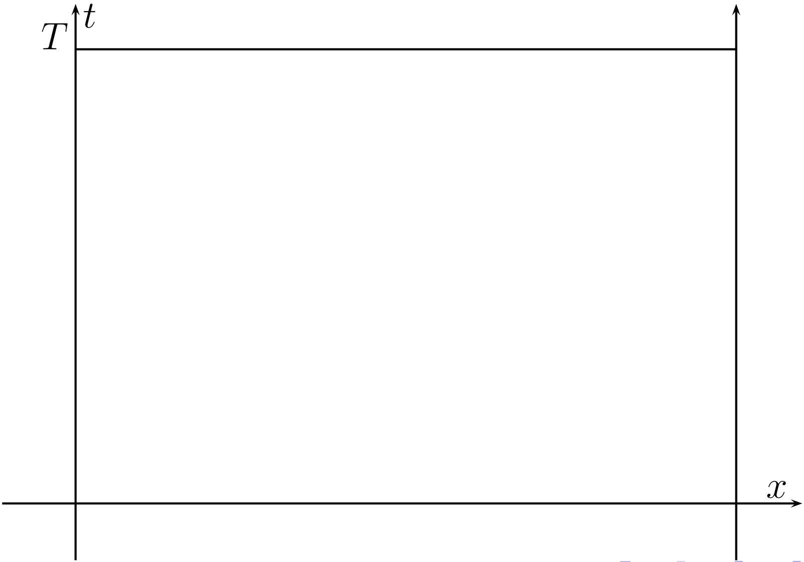
$$(2) \qquad T_1 > \frac{1}{2}\max\{L/|\Lambda_i(y^*)|;\, i \in \{1, \dots, n\}\},$$

$$(3) \qquad 2T_1 < T.$$

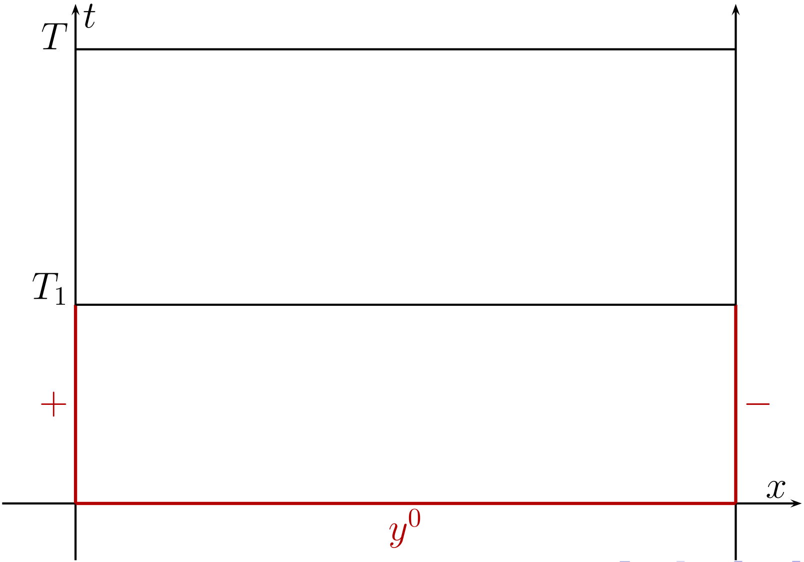
Let us recall that, for $y \in \mathbb{R}^n$, $y_- \in \mathbb{R}^m$ and $y_+ \in \mathbb{R}^{n-m}$ are such that

$$y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix}.$$

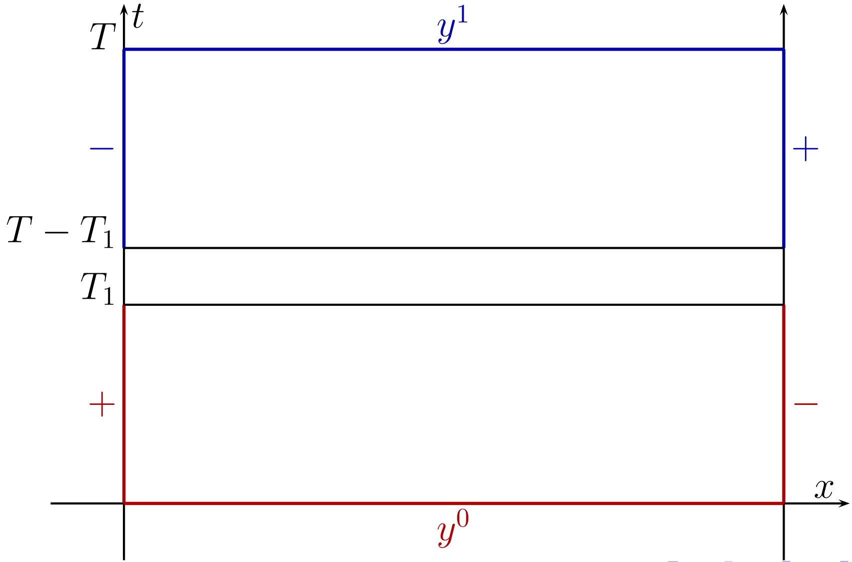
Sketch of proof (continued)



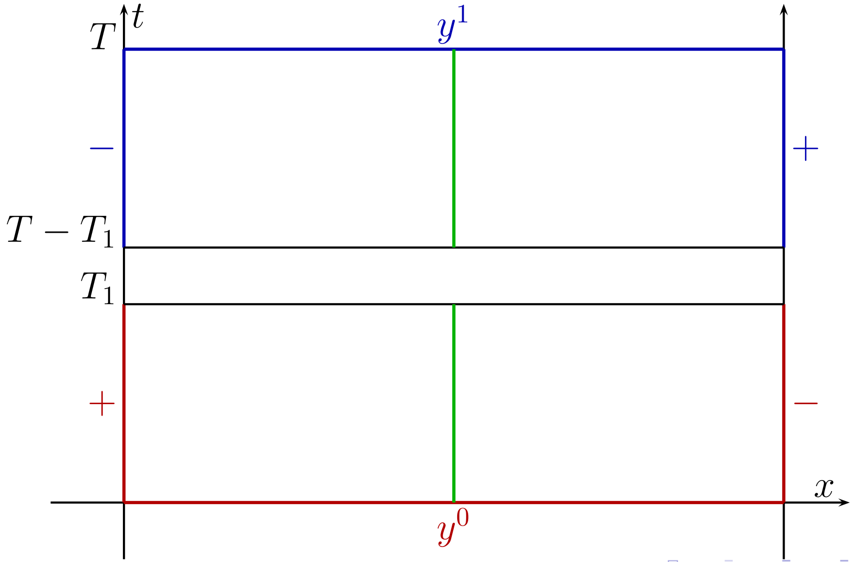
Sketch of proof (continued)



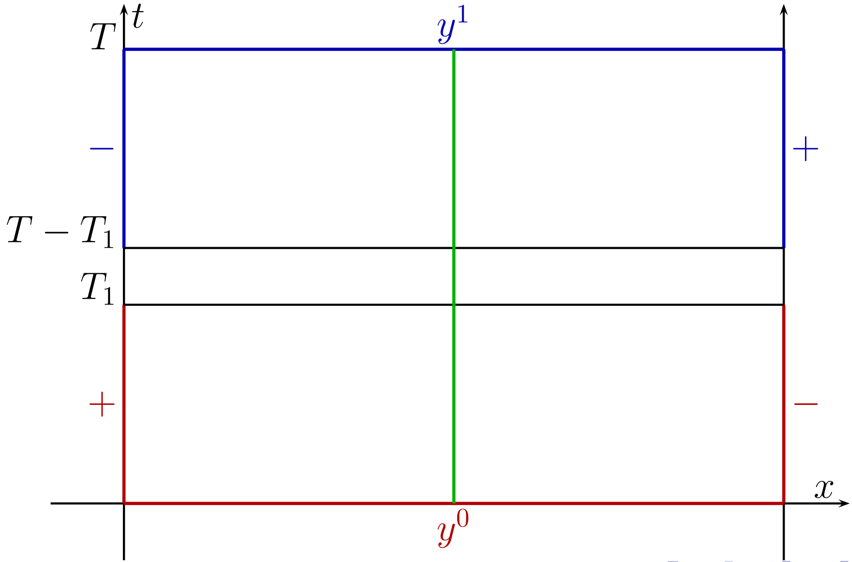
Sketch of proof (continued)



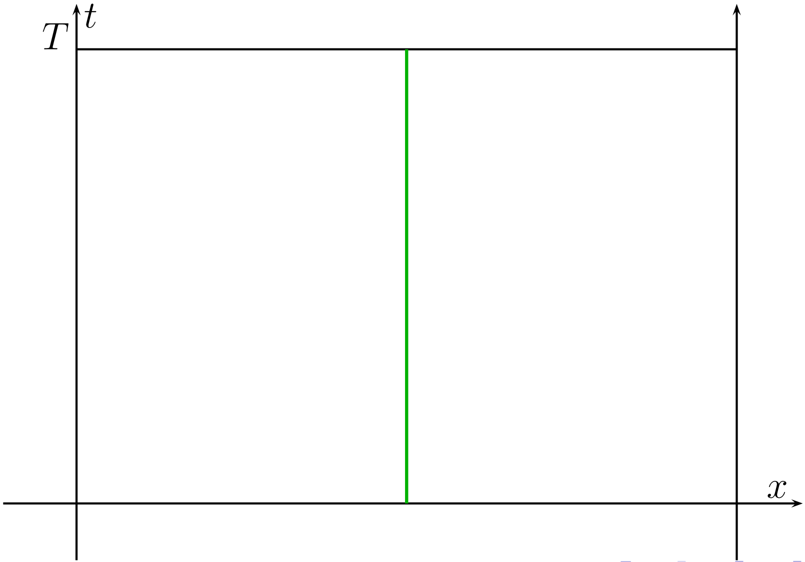
Sketch of proof (continued)



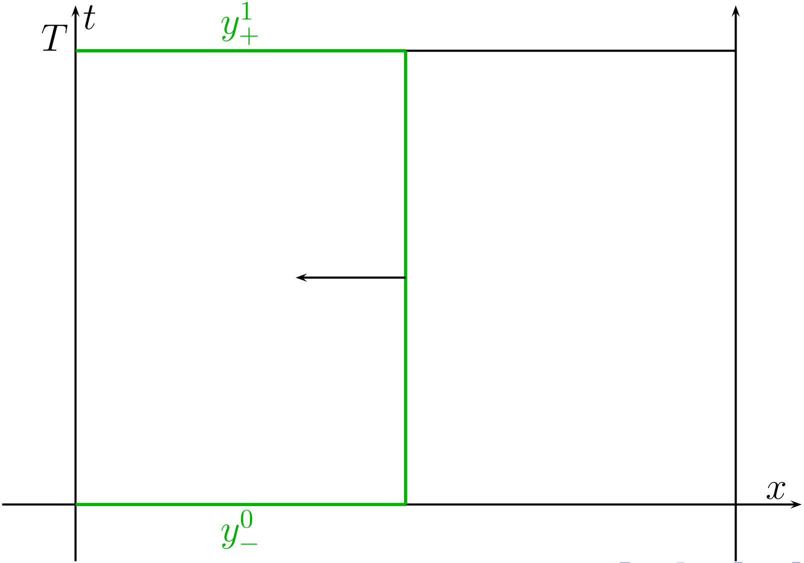
Sketch of proof (continued)



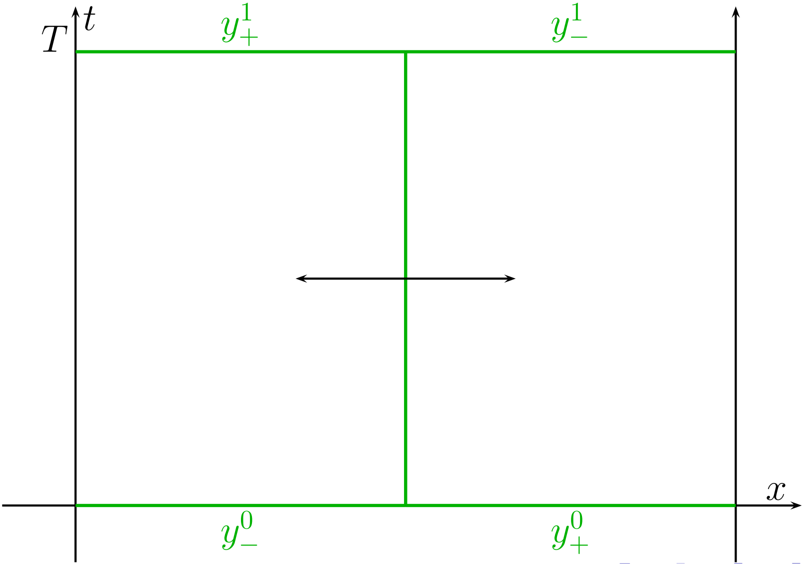
Sketch of proof (continued)



Sketch of proof (continued)

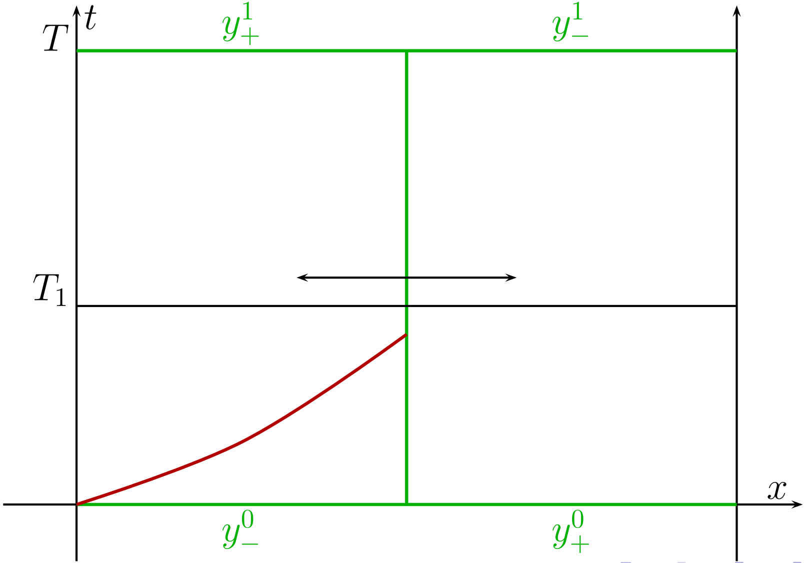


Sketch of proof (continued)

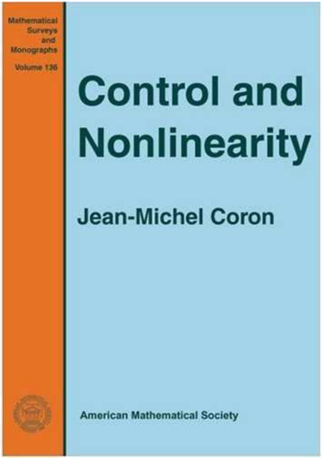


Sketch of proof (continued)

$T_1 > \max\{L/|\Lambda_i(0)|;\ i \in \{1, \ldots, n\}\}/2.$



More on controllability and stabilization



JMC, Control and nonlinearity, Mathematical Surveys and Monographs, 136, 2007, 427 p. Pdf file freely available from my web page.

Double inverted pendulum (CAS, ENSMP/La Villette)